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Padé approximant related to inequalities for Gauss lemniscate functions

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Abstract

Based on the Padé approximation method, we present new inequalities for Gauss lemniscate functions. We also solve a conjecture on inequalities for Gauss lemniscate functions proposed by Sun and Chen.

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1 Introduction

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points (x, y) in the plane satisfying the equation $(x^2 + y^2)^2 = x^2 + y^2$. In polar coordinates (r, θ), the equation becomes $r^2 = \cos(2\theta)$ and its arc length is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{1}{\sqrt{1-t^4}}\,\mathrm{d}t, \quad |x| \le 1, \tag{1.1}$$

where arcsl is called the arc lemniscate sine function studied by Gauss in 1797-1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{1.2}$$

The functions (1.1) and (1.2) can be found (see [1], Chapter 1, [2], p. 259, and [3–11]).

Another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh, have been introduced in [4], (3.1)-(3.2). Therein it has been proven that

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \quad x \in \mathbb{R} \tag{1.3}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \quad |x| < 1 \tag{1.4}$$

(see [4], Proposition 3.1).

Recently, numerous inequalities have been given for the lemniscate functions [6, 9–11]. For example, Neuman [6] proved the following inequalities:

$$\left(\frac{5}{3+2(1-x^4)^{1/2}}\right)^{1/2} < \frac{\operatorname{arcsl} x}{x} < \left(1-x^4\right)^{-1/10} \tag{1.5}$$

and

$$\left(\frac{5}{3+2(1+x^4)^{1/2}}\right)^{1/2} < \frac{\operatorname{arcslh} x}{x} < \left(1+x^4\right)^{-1/10} \tag{1.6}$$

for $0 < |x| < 1$.

Shafer [12] indicated several elementary quadratic approximations of selected functions without proof. Subsequently, Shafer [13] established these results as analytic inequalities. For example, Shafer [13] proved that, for $x > 0$,

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x. \tag{1.7}$$

The inequality (1.7) can also be found in [14]. Zhu [15] developed (1.7) to produce a symmetric double inequality

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x < \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}}, \quad x > 0, \tag{1.8}$$

where the constants $80/3$ and $256/\pi^2$ are the best possible. In [15], (1.8) is called a Shafer-type inequality.

Mortici and Srivastava [16] presented new bounds for $\arctan x$. Some inequalities for trigonometric functions were refined in [17].

Very recently, Sun and Chen [18] established the following Shafer-type inequalities for the lemniscate functions:

$$\frac{10}{5+\sqrt{25-10x^4}} < \frac{\operatorname{arcsl} x}{x}, \quad 0 < x < 1, \tag{1.9}$$

$$\frac{10}{5+\sqrt{25-15x^4}} < \frac{\operatorname{arctlh} x}{x}, \quad 0 < x < 1, \tag{1.10}$$

$$\frac{95}{80+\sqrt{225+285x^4}} < \frac{\operatorname{arcslh} x}{x}, \quad x > 0, \tag{1.11}$$

and presented the following conjecture.

Conjecture 1.1 *For $x > 0$,*

$$\frac{\operatorname{arcslh} x}{x} < \frac{95+\frac{931}{2925}x^{12}}{80+\sqrt{225+285x^4}} \tag{1.12}$$

and

$$\frac{1210}{940+9\sqrt{900+1210x^4}} < \frac{\operatorname{arctl} x}{x} < \frac{1210+\frac{2{,}078{,}417}{280{,}800}x^{12}}{940+9\sqrt{900+1210x^4}}. \tag{1.13}$$

Based on the Padé approximation method, in this paper we present new inequalities for Gauss lemniscate functions. We also prove Conjecture 1.1.

Some computations in this paper were performed using Maple software.

2 Padé approximant

For later use, we introduce the Padé approximant (see [19–21]). Let f be a formal power series,

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots. \tag{2.1}$$

The Padé approximation of order (p, q) of the function f is the rational function, denoted by

$$[p/q]_f(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j}, \tag{2.2}$$

where $p \geq 0$ and $q \geq 1$ are any given integers, the coefficients a_j and b_j are given by (see [19, 21])

$$\begin{cases} a_0 = c_0, \\ a_1 = c_0 b_1 + c_1, \\ a_2 = c_0 b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{2.3}$$

and we have

$$[p/q]_f(t) - f(t) = O\bigl(t^{p+q+1}\bigr). \tag{2.4}$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f. Moreover, we have (see [20])

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{2.5}$$

with $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, the nth partial sum of the series f (f_n is identically zero for $n < 0$).

Chen [9] presented the following power-series expansions (for $|x| < 1$):

$$\frac{\operatorname{arcsl} x}{x} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n}, \tag{2.6}$$

$$\frac{\operatorname{arcslh} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n}, \tag{2.7}$$

$$\frac{\operatorname{arctl} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!} x^{4n} \tag{2.8}$$

and

$$\frac{\operatorname{arctlh} x}{x} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!} x^{4n}. \tag{2.9}$$

We now consider the Padé approximant for the function $\frac{\operatorname{arcsl} x}{x}$ at the point $x = 0$. Let

$$f(t) = \sum_{j=0}^{\infty} c_j t^j = 1 + \frac{1}{10}t + \frac{1}{24}t^2 + \frac{5}{208}t^3 + \frac{35}{2176}t^4 + \cdots, \tag{2.10}$$

with the coefficients c_j given by

$$c_j = \frac{\Gamma(j+\frac{1}{2})}{\sqrt{\pi}(4j+1)\cdot j!}. \tag{2.11}$$

Let us find the $(2,2)$ Padé approximant for the function (2.10) at the point $t = 0$,

$$[2/2]_f(t) = \frac{\sum_{j=0}^{2} a_j t^j}{1 + \sum_{j=1}^{2} b_j t^j}.$$

Noting that

$$c_0 = 1, \qquad c_1 = \frac{1}{10}, \qquad c_2 = \frac{1}{24}, \qquad c_3 = \frac{5}{208}, \qquad c_4 = \frac{35}{2176}, \tag{2.12}$$

holds, we have, by (2.3),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 + \frac{1}{10}, \\ a_2 = b_2 + \frac{1}{10}b_1 + \frac{1}{24}, \\ 0 = \frac{5}{208} + \frac{1}{24}b_1 + \frac{1}{10}b_2, \\ 0 = \frac{35}{2176} + \frac{5}{508}b_1 + \frac{1}{24}b_2, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = -\frac{55}{68}, \qquad a_2 = \frac{23{,}623}{265{,}200}, \qquad b_1 = \frac{309}{340}, \qquad b_2 = \frac{489}{3536}.$$

We thus obtain

$$[2/2]_f(t) = \frac{1 - \frac{55}{68}t + \frac{23{,}623}{265{,}200}t^2}{1 - \frac{309}{340}t + \frac{489}{3536}t^2}, \tag{2.13}$$

and we have, by (2.4),

$$f(t) = [2/2]_f(t) + O\bigl(t^5\bigr). \tag{2.14}$$

That is

$$\sum_{j=0}^{\infty} \frac{\Gamma(j+\frac{1}{2})}{\sqrt{\pi}(4j+1)\cdot j!} t^j = \frac{1 - \frac{55}{68}t + \frac{23{,}623}{265{,}200}t^2}{1 - \frac{309}{340}t + \frac{489}{3536}t^2} + O\bigl(t^5\bigr). \tag{2.15}$$

Replacing t by x^4 in (2.15) yields

$$\begin{aligned} \frac{\operatorname{arcsl} x}{x} &= \frac{1 - \frac{55}{68}x^4 + \frac{23{,}623}{265{,}200}x^8}{1 - \frac{309}{340}x^4 + \frac{489}{3536}x^8} + O\bigl(x^{20}\bigr) \\ &= \frac{265{,}200 - 214{,}500x^4 + 23{,}623x^8}{15(17{,}680 - 16{,}068x^4 + 2445x^8)} + O\bigl(x^{20}\bigr). \end{aligned} \tag{2.16}$$

Remark 2.1 *Using* (2.5), *we can also derive* (2.13). *Indeed, we have*

$$\begin{aligned} [2/2]_f(t) &= \frac{\begin{vmatrix} t^2 f_0(t) & t f_1(t) & f_2(t) \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}}{\begin{vmatrix} t^2 & t & 1 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} t^2 & t(1+\frac{1}{10}t) & 1 + \frac{1}{10}t + \frac{1}{24}t^2 \\ \frac{1}{10} & \frac{1}{24} & \frac{5}{208} \\ \frac{1}{24} & \frac{5}{208} & \frac{35}{2176} \end{vmatrix}}{\begin{vmatrix} t^2 & t & 1 \\ \frac{1}{10} & \frac{1}{24} & \frac{5}{208} \\ \frac{1}{24} & \frac{5}{208} & \frac{35}{2176} \end{vmatrix}} \\ &= \frac{1 - \frac{55}{68}t + \frac{23{,}623}{265{,}200}t^2}{1 - \frac{309}{340}t + \frac{489}{3536}t^2}. \end{aligned}$$

Following the same method as used in the derivation of the formula (2.16), we find

$$\begin{aligned} \frac{\operatorname{arcslh} x}{x} &= \frac{1 + \frac{55}{68}x^4 + \frac{23{,}623}{265{,}200}x^8}{1 + \frac{309}{340}x^4 + \frac{489}{3536}x^8} + O\bigl(x^{20}\bigr) \\ &= \frac{265{,}200 + 214{,}500x^4 + 23{,}623x^8}{15(17{,}680 + 16{,}068x^4 + 2445x^8)} + O\bigl(x^{20}\bigr), \end{aligned} \tag{2.17}$$

$$\frac{\operatorname{arctl} x}{x} = \frac{1 + \frac{63}{130}x^4 - \frac{139}{6240}x^8}{1 + \frac{33}{52}x^4} + O\bigl(x^{16}\bigr), \tag{2.18}$$

$$\begin{aligned} \frac{\operatorname{arctl} x}{x} &= \frac{1 + \frac{79{,}047}{94{,}520}x^4 + \frac{565{,}795}{5{,}898{,}048}x^8}{1 + \frac{18{,}645}{18{,}904}x^4 + \frac{336{,}105}{1{,}966{,}016}x^8} + O\bigl(x^{20}\bigr) \\ &= \frac{29{,}490{,}240 + 24{,}662{,}664x^4 + 2{,}828{,}975x^8}{15(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)} + O\bigl(x^{20}\bigr) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\frac{\operatorname{arctlh} x}{x} &= \frac{1 - \frac{79,047}{94,520}x^4 + \frac{565,795}{5,898,048}x^8}{1 - \frac{18,645}{18,904}x^4 + \frac{336,105}{1,966,016}x^8} + O\bigl(x^{20}\bigr) \\ &= \frac{29,490,240 - 24,662,664x^4 + 2,828,975x^8}{15(1,966,016 - 1,939,080x^4 + 336,105x^8)} + O\bigl(x^{20}\bigr).\end{aligned} \tag{2.20}$$

In view of (2.16) and (2.17), we pose the following.

Conjecture 2.1 *Let*

$$\frac{\operatorname{arcsl} x}{x} = \frac{1 + \sum_{j=1}^{n} a_j x^{4j}}{1 + \sum_{j=1}^{n} b_j x^{4j}} + O\bigl(x^{8n+4}\bigr) \tag{2.21}$$

and

$$\frac{\operatorname{arcslh} x}{x} = \frac{1 + \sum_{j=1}^{n} \alpha_j x^{4j}}{1 + \sum_{j=1}^{n} \beta_j x^{4j}} + O\bigl(x^{8n+4}\bigr). \tag{2.22}$$

Then the coefficients a_j and α_j satisfy the following relation:

$$a_j = (-1)^j \alpha_j, \quad j = 1, 2, \ldots, n, \tag{2.23}$$

and the coefficients b_j and β_j satisfy the following relation:

$$b_j = (-1)^j \beta_j, \quad j = 1, 2, \ldots, n. \tag{2.24}$$

In view of (2.19) and (2.20), we pose the following.

Conjecture 2.2 *Let*

$$\frac{\operatorname{arctl} x}{x} = \frac{1 + \sum_{j=1}^{n} p_j x^{4j}}{1 + \sum_{j=1}^{n} q_j x^{4j}} + O\bigl(x^{8n+4}\bigr) \tag{2.25}$$

and

$$\frac{\operatorname{arctlh} x}{x} = \frac{1 + \sum_{j=1}^{n} r_j x^{4j}}{1 + \sum_{j=1}^{n} s_j x^{4j}} + O\bigl(x^{8n+4}\bigr). \tag{2.26}$$

Then the coefficients p_j and r_j satisfy the following relation:

$$p_j = (-1)^j r_j, \quad j = 1, 2, \ldots, n, \tag{2.27}$$

and the coefficients q_j and s_j satisfy the following relation:

$$q_j = (-1)^j s_j, \quad j = 1, 2, \ldots, n. \tag{2.28}$$

3 Inequalities

Equations (2.16)-(2.20) motivate us to establish the following theorems.

Theorem 3.1 *For* $0 < x < 1$,

$$\frac{265{,}200 - 214{,}500x^4 + 23{,}623x^8}{15(17{,}680 - 16{,}068x^4 + 2445x^8)} < \frac{\operatorname{arcsl} x}{x}. \tag{3.1}$$

Proof Consider the function

$$f(x) = \operatorname{arcsl} x - \frac{x(265{,}200 - 214{,}500x^4 + 23{,}623x^8)}{15(17{,}680 - 16{,}068x^4 + 2445x^8)}, \quad 0 < x < 1.$$

Differentiation yields

$$\begin{aligned} f'(x) = {} & \frac{1}{\sqrt{1-x^4}} \\ & - \frac{312{,}582{,}400 - 411{,}873{,}280x^4 + 177{,}771{,}984x^8 - 21{,}634{,}288x^{12} + 3{,}850{,}549x^{16}}{(17{,}680 - 16{,}068x^4 + 2445x^8)^2}. \end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} & \left(\frac{1}{\sqrt{1-t}}\right)^2 \\ & \quad - \left(\frac{312{,}582{,}400 - 411{,}873{,}280t + 177{,}771{,}984t^2 - 21{,}634{,}288t^3 + 3{,}850{,}549t^4}{(17{,}680 - 16{,}068t + 2445t^2)^2}\right)^2 \\ & = \frac{t^5 g(t)}{(1-t)(17{,}680 - 16{,}068t + 2445t^2)^4}, \quad 0 < t < 1, \end{aligned}$$

where

$$\begin{aligned} g(t) = {} & 1{,}744{,}280{,}123{,}040{,}000 - 2{,}406{,}774{,}938{,}256{,}000t \\ & + 1{,}064{,}272{,}682{,}007{,}600t^2 \\ & - 145{,}697{,}716{,}749{,}000t^3 + 14{,}826{,}727{,}601{,}401t^4. \end{aligned}$$

We now prove that $f'(x) > 0$ for $0 < x < 1$. It suffices to show that $g(t) > 0$ for $0 < t < 1$. Differentiation yields

$$\begin{aligned} g'(t) = {} & -2{,}406{,}774{,}938{,}256{,}000 + 2{,}128{,}545{,}364{,}015{,}200t \\ & - 437{,}093{,}150{,}247{,}000t^2 \\ & + 59{,}306{,}910{,}405{,}604t^3 \end{aligned}$$

and

$$\begin{aligned} g''(t) = {} & 2{,}128{,}545{,}364{,}015{,}200 - 874{,}186{,}300{,}494{,}000t \\ & + 177{,}920{,}731{,}216{,}812t^2 > 0, \quad 0 < t < 1. \end{aligned}$$

We then obtain, for $0 < t < 1$,

$$\begin{aligned} &g'(t) < g'(1) = -656{,}015{,}814{,}082{,}196 < 0 \quad \Longrightarrow \\ &g(t) > g(1) = 270{,}906{,}877{,}644{,}001 > 0. \end{aligned}$$

Hence, $f'(x) > 0$ for $0 < x < 1$, and we have

$$f(x) > f(0) = 0, \quad 0 < x < 1.$$

The proof is complete. □

Remark 3.1 *There is no strict comparison between the two lower bounds in* (1.5) *and* (3.1).

Theorem 3.2 *For* $x > 0$,

$$\frac{\operatorname{arcslh} x}{x} < \frac{265{,}200 + 214{,}500x^4 + 23{,}623x^8}{15(17{,}680 + 16{,}068x^4 + 2445x^8)}. \tag{3.2}$$

Proof Consider the function

$$F(x) = \operatorname{arcslh} x - \frac{x(265{,}200 + 214{,}500x^4 + 23{,}623x^8)}{15(17{,}680 + 16{,}068x^4 + 2445x^8)}, \quad x > 0.$$

Differentiation yields

$$\begin{aligned} F'(x) = {}& \frac{1}{\sqrt{1+x^4}} \\ & - \frac{312{,}582{,}400 + 411{,}873{,}280x^4 + 177{,}771{,}984x^8 + 21{,}634{,}288x^{12} + 3{,}850{,}549x^{16}}{(17{,}680 + 16{,}068x^4 + 2445x^8)^2}. \end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} &\left(\frac{1}{\sqrt{1+t}}\right)^2 \\ &\quad - \left(\frac{312{,}582{,}400 + 411{,}873{,}280t + 177{,}771{,}984t^2 + 21{,}634{,}288t^3 + 3{,}850{,}549t^4}{(17{,}680 + 16{,}068t + 2445t^2)^2}\right)^2 \\ &= -\frac{t^5 G(t)}{(1+t)(17{,}680 + 16{,}068t + 2445t^2)^4}, \end{aligned}$$

where

$$\begin{aligned} G(t) = {}& 1{,}744{,}280{,}123{,}040{,}000 + 2{,}406{,}774{,}938{,}256{,}000t + 1{,}064{,}272{,}682{,}007{,}600t^2 \\ & + 145{,}697{,}716{,}749{,}000t^3 + 14{,}826{,}727{,}601{,}401t^4. \end{aligned}$$

Hence, $F'(x) < 0$ for $x > 0$, and we have

$$F(x) < F(0) = 0, \quad x > 0.$$

The proof is complete. □

Remark 3.2 *For* $0 < t < 1$, *we find*

$$I(t) := \frac{1}{1+t} - \left(\frac{265{,}200 + 214{,}500t + 23{,}623t^2}{15(17{,}680 + 16{,}068t + 2445t^2)}\right)^{10} = \frac{t^2 P_{19}(t)}{576{,}650{,}390{,}625(1+t)(17{,}680 + 16{,}068t + 2445t^2)^{10}}$$

with

$$P_{19}(t) = P_{16}(t) + t^{17} P_2(t),$$

where

$$\begin{aligned} P_{16}(t) = {} & 3{,}309{,}224{,}024{,}069{,}080{,}418{,}989{,}754{,}522{,}912{,}085{,}339{,}870{,}997{,}824{,}000t^{16} \\ & + \cdots \\ & + 229{,}442{,}535{,}851{,}108{,}636{,}620{,}015{,}850{,}036{,}920{,}320{,}000{,}000{,}000{,}000{,}000{,}000 \end{aligned}$$

is a polynomial of the 16*th degree, having all coefficients positive, and*

$$\begin{aligned} P_2(t) = {} & 77{,}541{,}624{,}086{,}159{,}498{,}428{,}020{,}328{,}992{,}837{,}339{,}887{,}447{,}064{,}000 \\ & - 565{,}686{,}157{,}207{,}722{,}134{,}655{,}870{,}693{,}904{,}642{,}976{,}763{,}301{,}024t \\ & - 54{,}119{,}091{,}759{,}561{,}776{,}058{,}592{,}767{,}712{,}571{,}305{,}215{,}681{,}649t^2 > 0 \end{aligned}$$

for $0 < t < 1$. *So*, $I(t) > 0$ *for* $0 < t < 1$. *We then see that the inequality* (3.2) *is sharper than the right side of* (1.6).

Theorem 3.3 *For* $x > 0$,

$$\frac{1 + \frac{63}{130}x^4 - \frac{139}{6240}x^8}{1 + \frac{33}{52}x^4} < \frac{\operatorname{arctl} x}{x} < \frac{29{,}490{,}240 + 24{,}662{,}664x^4 + 2{,}828{,}975x^8}{15(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)}. \tag{3.3}$$

Proof Consider the function

$$\lambda(x) = \operatorname{arctl} x - \frac{x(1 + \frac{63}{130}x^4 - \frac{139}{6240}x^8)}{1 + \frac{33}{52}x^4}.$$

Differentiation yields

$$\lambda'(x) = \frac{1}{(1+x^4)^{3/4}} + \frac{x^3(48{,}672 + 14{,}456x^4 + 4587x^8)}{30(52 + 33x^4)^2} > 0.$$

We then obtain

$$\lambda(x) > \lambda(0) = 0, \quad x > 0.$$

Hence the first inequality in (3.3) holds for $x > 0$.

Consider the function

$$T(x) = \operatorname{arctl} x - \frac{x(29{,}490{,}240 + 24{,}662{,}664x^4 + 2{,}828{,}975x^8)}{15(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)}, \quad x > 0.$$

Differentiation yields

$$T'(x) = \frac{1}{(1+x^4)^{3/4}} - \frac{P_{16}(x)}{(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)^2},$$

where

$$\begin{aligned} P_{16}(x) ={}& 3{,}865{,}218{,}912{,}256 + 4{,}725{,}610{,}426{,}368x^4 + 1{,}899{,}763{,}315{,}008x^8 \\ &+ 170{,}687{,}344{,}256x^{12} + 63{,}388{,}842{,}825x^{16}. \end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} &\frac{1}{(1+x^4)^3} - \left(\frac{P_{16}(x)}{(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)^2}\right)^4 \\ &\quad = -\frac{x^{20}P_{56}(x)}{(1+x^4)^3(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)^8}, \end{aligned}$$

where

$$\begin{aligned} P_{56}(x) ={}& 13{,}193{,}567{,}461{,}486{,}862{,}074{,}082{,}196{,}527{,}146{,}063{,}235{,}598{,}765{,}785{,}088 \\ &+ 75{,}159{,}817{,}580{,}420{,}162{,}914{,}879{,}309{,}165{,}363{,}929{,}497{,}102{,}849{,}146{,}880x^4 \\ &+ 190{,}493{,}075{,}741{,}254{,}897{,}950{,}338{,}074{,}805{,}626{,}536{,}902{,}462{,}936{,}186{,}880x^8 \\ &+ 283{,}233{,}781{,}637{,}227{,}052{,}608{,}425{,}496{,}608{,}925{,}420{,}321{,}925{,}549{,}260{,}800x^{12} \\ &+ 274{,}381{,}791{,}750{,}085{,}496{,}947{,}276{,}941{,}731{,}670{,}794{,}946{,}132{,}888{,}780{,}800x^{16} \\ &+ 182{,}271{,}787{,}590{,}701{,}615{,}339{,}301{,}540{,}193{,}194{,}594{,}557{,}235{,}390{,}578{,}688x^{20} \\ &+ 85{,}570{,}287{,}614{,}566{,}775{,}085{,}144{,}063{,}641{,}805{,}696{,}286{,}360{,}924{,}323{,}840x^{24} \\ &+ 29{,}163{,}131{,}006{,}055{,}200{,}534{,}183{,}374{,}987{,}447{,}946{,}919{,}333{,}657{,}968{,}640x^{28} \\ &+ 7{,}481{,}144{,}367{,}677{,}341{,}229{,}619{,}045{,}201{,}182{,}337{,}247{,}982{,}930{,}534{,}400x^{32} \\ &+ 1{,}502{,}545{,}339{,}351{,}309{,}468{,}552{,}186{,}115{,}563{,}901{,}082{,}330{,}882{,}201{,}600x^{36} \\ &+ 238{,}495{,}639{,}577{,}137{,}257{,}561{,}813{,}891{,}822{,}862{,}592{,}696{,}929{,}928{,}896x^{40} \\ &+ 29{,}999{,}531{,}147{,}567{,}967{,}753{,}948{,}099{,}263{,}441{,}234{,}315{,}939{,}560{,}800x^{44} \\ &+ 3{,}208{,}050{,}013{,}558{,}968{,}652{,}633{,}219{,}730{,}412{,}159{,}840{,}683{,}611{,}875x^{48} \\ &+ 222{,}336{,}558{,}000{,}169{,}152{,}230{,}844{,}556{,}985{,}454{,}831{,}178{,}171{,}875x^{52} \\ &+ 16{,}145{,}492{,}412{,}888{,}980{,}411{,}169{,}048{,}998{,}579{,}532{,}875{,}390{,}625x^{56}. \end{aligned}$$

Hence, $T'(x) < 0$ for $x > 0$, and we have

$$T(x) < T(0) = 0, \quad x > 0.$$

Hence, the second inequality in (3.3) holds for $x > 0$. The proof is complete. □

Theorem 3.4 *For* $0 < x < 1$,

$$\frac{29{,}490{,}240 - 24{,}662{,}664x^4 + 2{,}828{,}975x^8}{15(1{,}966{,}016 - 1{,}939{,}080x^4 + 336{,}105x^8)} < \frac{\operatorname{arctlh} x}{x}. \tag{3.4}$$

Proof Consider the function

$$H(x) = \operatorname{arctlh} x - \frac{x(29{,}490{,}240 - 24{,}662{,}664x^4 + 2{,}828{,}975x^8)}{15(1{,}966{,}016 - 1{,}939{,}080x^4 + 336{,}105x^8)}, \quad 0 < x < 1.$$

Differentiation yields

$$H'(x) = \frac{1}{(1-x^4)^{3/4}} - \frac{Q(x^4)}{(1{,}966{,}016 - 1{,}939{,}080x^4 + 336{,}105x^8)^2},$$

where

$$\begin{aligned} Q(t) = {}& 3{,}865{,}218{,}912{,}256 - 4{,}725{,}610{,}426{,}368t + 1{,}899{,}763{,}315{,}008t^2 \\ & - 170{,}687{,}344{,}256t^3 + 63{,}388{,}842{,}825t^4, \quad 0 < t < 1. \end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} & \frac{1}{(1-t)^3} - \left(\frac{Q(t)}{(1{,}966{,}016 - 1{,}939{,}080t + 336{,}105t^2)^2}\right)^4 \\ & \quad = \frac{t^5 R(t)}{(1-t)^3(1{,}966{,}016 - 1{,}939{,}080t + 336{,}105t^2)^8}, \end{aligned}$$

where

$$\begin{aligned} R(t) = {}& 301{,}748{,}693{,}573{,}399{,}407{,}094{,}173{,}717{,}482{,}883{,}533{,}487{,}653{,}121 \\ & + 9{,}932{,}615{,}535{,}811{,}554{,}413{,}076{,}061{,}553{,}884{,}665{,}646{,}471{,}005{,}365(1-t) \\ & + 151{,}766{,}449{,}766{,}787{,}034{,}704{,}161{,}483{,}173{,}419{,}604{,}277{,}111{,}004{,}855(1-t)^2 \\ & + 680{,}433{,}563{,}535{,}649{,}162{,}659{,}902{,}808{,}405{,}182{,}093{,}255{,}793{,}778{,}810(1-t)^3 \\ & + 1856{,}390{,}570{,}444{,}186{,}005{,}047{,}799{,}006{,}039{,}729{,}435{,}155{,}242{,}856{,}245(1-t)^4 \\ & + 3{,}124{,}371{,}679{,}128{,}783{,}209{,}196{,}299{,}976{,}215{,}123{,}075{,}149{,}799{,}026{,}971(1-t)^5 \\ & + 3{,}542{,}098{,}875{,}374{,}455{,}780{,}446{,}672{,}503{,}755{,}643{,}630{,}859{,}189{,}032{,}595(1-t)^6 \\ & + 2{,}471{,}319{,}403{,}553{,}122{,}128{,}066{,}406{,}333{,}619{,}269{,}039{,}662{,}418{,}255{,}020(1-t)^7 \\ & + 1{,}090{,}939{,}975{,}419{,}395{,}315{,}836{,}646{,}390{,}524{,}713{,}606{,}873{,}688{,}710{,}195(1-t)^8 \\ & + 188{,}439{,}516{,}872{,}719{,}220{,}883{,}777{,}738{,}283{,}311{,}857{,}263{,}725{,}003{,}515(1-t)^9 \\ & + 72{,}805{,}480{,}166{,}035{,}035{,}195{,}735{,}976{,}881{,}949{,}595{,}398{,}022{,}003{,}221(1-t)^{10} \\ & + 2{,}968{,}223{,}270{,}581{,}948{,}926{,}689{,}804{,}107{,}877{,}843{,}092{,}991{,}437{,}050(1-t)^{11} \\ & + \bigl(1{,}786{,}914{,}569{,}129{,}666{,}891{,}048{,}623{,}948{,}471{,}984{,}527{,}027{,}924{,}375 \\ & - 3{,}700{,}335{,}780{,}276{,}573{,}525{,}522{,}128{,}994{,}658{,}629{,}077{,}296{,}875(1-t)\bigr)(1-t)^{12} \\ & + 16{,}145{,}492{,}412{,}888{,}980{,}411{,}169{,}048{,}998{,}579{,}532{,}875{,}390{,}625(1-t)^{14}. \end{aligned}$$

Since $R(t) > 0$ for $0 < t < 1$, we have $H'(x) > 0$ for $0 < x < 1$. We then obtain

$$H(x) > H(0) = 0, \quad 0 < x < 1.$$

The proof is complete. □

4 Proof of Conjecture 1.1

Proof of (1.12) It suffices to show by (3.2) that

$$\frac{265{,}200 + 214{,}500x^4 + 23{,}623x^8}{15(17{,}680 + 16{,}068x^4 + 2445x^8)} < \frac{95 + \frac{931}{2925}x^{12}}{80 + \sqrt{225 + 285x^4}}, \quad x > 0,$$

i.e.,

$$\frac{95 + \frac{931}{2925}x^{12}}{\frac{265{,}200+214{,}500x^4+23{,}623x^8}{15(17{,}680+16{,}068x^4+2445x^8)}} - 80 > \sqrt{225 + 285x^4}, \quad x > 0. \tag{4.1}$$

Elementary calculations show that

$$\begin{aligned}&\left(\frac{95 + \frac{931}{2925}x^{12}}{\frac{265{,}200+214{,}500x^4+23{,}623x^8}{15(17{,}680+16{,}068x^4+2445x^8)}} - 80\right)^2 - \left(\sqrt{225 + 285x^4}\right)^2 \\ &\quad = \frac{x^{16}P_{24}(x)}{(265{,}200 + 214{,}500x^4 + 23{,}623x^8)^2},\end{aligned}$$

where

$$\begin{aligned}P_{24}(x) &= 2{,}214{,}994{,}396{,}680{,}000 + 2{,}167{,}794{,}625{,}751{,}625x^4 \\ &\quad + 771{,}850{,}648{,}332{,}400x^8 \\ &\quad + 100{,}410{,}388{,}038{,}870x^{12} + 15{,}721{,}941{,}655{,}056x^{16} + 3{,}584{,}399{,}789{,}880x^{20} \\ &\quad + 272{,}711{,}522{,}475x^{24}.\end{aligned}$$

We see from $P_{24}(x) > 0$ that (4.1) holds. The proof is complete. □

Proof of (1.13) First of all, we prove the second inequality in (1.13). It suffices to show by the right-hand side of (3.3) that

$$\frac{29{,}490{,}240 + 24{,}662{,}664x^4 + 2{,}828{,}975x^8}{15(1{,}966{,}016 + 1{,}939{,}080x^4 + 336{,}105x^8)} < \frac{1210 + \frac{2{,}078{,}417}{280{,}800}x^{12}}{940 + 9\sqrt{900 + 1210x^4}}, \quad x > 0,$$

i.e.,

$$\frac{1210 + \frac{2{,}078{,}417}{280{,}800}x^{12}}{\frac{29{,}490{,}240+24{,}662{,}664x^4+2{,}828{,}975x^8}{15(1{,}966{,}016+1{,}939{,}080x^4+336{,}105x^8)}} - 940 > 9\sqrt{900 + 1210x^4}, \quad x > 0. \tag{4.2}$$

Elementary calculations show that

$$\left(\frac{1210+\frac{2,078,417}{280,800}x^{12}}{\frac{29,490,240+24,662,664x^4+2,828,975x^8}{15(1,966,016+1,939,080x^4+336,105x^8)}}-940\right)^2-\left(9\sqrt{900+1210x^4}\right)^2$$
$$=\frac{P_{40}(x)}{350,438,400(29,490,240+24,662,664x^4+2,828,975x^8)^2},$$

where

$$\begin{aligned}P_{40}(x)={}&423,992,204,507,234,653,175,808,000,000\\&+813,161,362,018,201,812,231,782,400,000x^4\\&+516,869,957,972,853,387,975,720,960,000x^8\\&+125,747,707,439,033,797,403,639,808,000x^{12}\\&+18,812,522,309,950,882,139,627,520,000x^{16}\\&+6,902,175,873,182,801,970,021,120,000x^{20}\\&+1,857,663,393,190,652,946,224,195,584x^{24}\\&+192,485,925,752,231,924,989,587,840x^{28}\\&+21,951,612,856,626,590,316,104,640x^{32}\\&+5,630,747,696,194,377,041,485,200x^{36}\\&+487,994,939,463,408,187,266,225x^{40}.\end{aligned}$$

We see from $P_{40}(x)>0$ that (4.2) holds. Hence, the second inequality in (1.13) holds.

Second, we prove the first inequality in (1.13). We consider two cases.

Case 1. $0<x<1$.

It suffices to show by the left-hand side of (3.3) that

$$\frac{1210}{940+9\sqrt{900+1210x^4}}<\frac{1+\frac{63}{130}x^4-\frac{139}{6240}x^8}{1+\frac{33}{52}x^4},\quad 0<x<1,$$

i.e.,

$$9\sqrt{900+1210x^4}>\frac{1210}{\frac{1+\frac{63}{130}x^4-\frac{139}{6240}x^8}{1+\frac{33}{52}x^4}}-940,\quad 0<x<1. \tag{4.3}$$

Elementary calculations show that

$$\left(9\sqrt{900+1210x^4}\right)^2-\left(\frac{1210}{\frac{1+\frac{63}{130}x^4-\frac{139}{6240}x^8}{1+\frac{33}{52}x^4}}-940\right)^2$$
$$=\frac{1210x^{12}(128,621,376-81,039,502x^4+1,565,001x^8)}{(6240+3024x^4-139x^8)^2}>0,\quad 0<x<1,$$

which shows that (4.3) holds.

Case 2. $x \geq 1$.

Consider the function $U(x)$ defined by

$$U(x) = \operatorname{arctl} x - \frac{1210x}{940 + 9\sqrt{900 + 1210x^4}}.$$

Differentiation yields

$$U'(x) = \frac{1}{(1+x^4)^{3/4}} + \frac{12{,}100(1089x^4 - 94\sqrt{900 + 1210x^4} - 810)}{(940 + 9\sqrt{900 + 1210x^4})^2\sqrt{900 + 1210x^4}}. \tag{4.4}$$

Noting that

$$1089x^4 - 94\sqrt{900 + 1210x^4} - 810 > 0, \quad x \geq 2,$$

holds, we obtain

$$U'(x) > 0, \quad x \geq 2.$$

We now show that $U'(x) > 0$ is also valid for $1 \leq x < 2$. It suffices to show that

$$y(x) > 0, \quad 1 \leq x < 2,$$

where

$$y(x) = y_1(x) + y_2(x),$$

with

$$y_1(x) = \frac{(940 + 9\sqrt{900 + 1210x^4})^2\sqrt{900 + 1210x^4}}{12{,}100(1+x^4)^{3/4}} + 1089x^4 - 810$$

and

$$y_2(x) = -94\sqrt{900 + 1210x^4}.$$

Differentiation yields

$$y_1'(x) = \frac{x^3(940 + 9\sqrt{900 + 1210x^4})y_3(x)}{1210(1+x^4)^{7/4}\sqrt{900 + 1210x^4}} + 32{,}596x^3,$$

where

$$\begin{aligned} y_3(x) &= 4104\sqrt{900 + 1210x^4} + 3267x^4\sqrt{900 + 1210x^4} - 113{,}740x^4 - 26{,}320 \\ &> 4104\sqrt{1210x^4} + 3267x^4\sqrt{1210x^4} - 113{,}740x^4 - 26{,}320 \\ &= 81{,}081\sqrt{10} - 140{,}060 + (305{,}910\sqrt{10} - 454{,}960)(x-1) \\ &\quad + (584{,}199\sqrt{10} - 682{,}440)(x-1)^2 + (718{,}740\sqrt{10} - 454{,}960)(x-1)^3 \end{aligned}$$

$$+ (539{,}055\sqrt{10} - 113{,}740)(x-1)^4 + 215{,}622\sqrt{10}(x-1)^5 + 35{,}937\sqrt{10}(x-1)^6$$
$$> 0 \quad \text{for } 1 \le x < 2.$$

Hence, we have $y_1'(x) > 0$ for $1 \le x < 2$.

Let $1 \le r \le x \le s \le 2$. Since $y_1(x)$ is increasing and $y_2(x)$ is decreasing for $1 \le x \le 2$, we obtain

$$y(x) \ge y_1(r) + y_2(s) =: \sigma_1(r,s).$$

We divide the interval $[1,2]$ into 100 subintervals:

$$[1,2] = \bigcup_{k=0}^{99}\left[1 + \frac{k}{100}, 1 + \frac{k+1}{100}\right] > 0 \quad \text{for } k = 0,1,2,\ldots,99.$$

By direct computation we get

$$\sigma_1\left(1 + \frac{k}{100}, 1 + \frac{k+1}{100}\right) > 0 \quad \text{for } k = 0,1,2,\ldots,99.$$

Hence,

$$y(x) > 0 \quad \text{for } x \in \left[1 + \frac{k}{100}, 1 + \frac{k+1}{100}\right] \text{ and } k = 0,1,2,\ldots,99.$$

This proves $U'(x) > 0$ for $1 \le x < 2$.

We then obtain $U'(x) > 0$ for all $x \ge 1$, and we have

$$U(x) > U(1) = 0.00154438\ldots > 0 \quad \text{for } x \ge 1,$$

which shows the first inequality in (1.13) holds for $x \ge 1$. Thus, the first inequality in (1.13) holds for all $x > 0$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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